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HEAT- AND MASS-TRANSFER CALCULATIONS USING AN EXPONENTIALLY CURVED EQUILIBRIUM LINE WITH SPECIAL REFERENCE TO COOLING TOWERS

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Abstract—Simple formulae and charts are already available to calculate the performance of heat- and mass-transfer equipment if the relation expressing equilibrium between the phases is linear and the transfer coefficient is given.

Examples are gas absorbers if Henry's law is obeyed or heat exchangers transferring sensible heat.

It is shown that simple charts can also be prepared where the equilibrium relation can be expressed in the form

$$u^* = A + Bv + C\exp(Dv)$$

where u^* is a potential in equilibrium with potential v in another phase and A, B, C and D are constants. A limited number of charts will cover all conditions.

The situations examined are parallel flow of phases—both concurrent and countercurrent—crossflow of the phases and the time-dependent case where one phase is stationary and the other moves through it.

The exponential equilibrium relation can be empirically fitted with high accuracy to many industrially important classes of equilibrium data over wide ranges.

NOMENCLATURE

A, B, C, D,	constants in equilibrium relation;
c_L,	specific heat of water;
G,	inert gas mass flow rate;
h,	enthalpy of moist air per unit mass of dry air;
h_0,	constant value of h;
h^*,	value of h for air in equilibrium with water at temperature T;
H,	defined by equations (8);
H_0,	defined by equation (9);
H_{00},	value of H_0 when $R_0 = \pm 1$;
I,	N.T.U. integral [equation (6)];
k,	transfer coefficient;
L,	water mass flow rate;
m,	mass of adsorbate per unit mass of inert gas;
m_0,	constant value of m;
m^*,	value of m for gas in equilibrium with adsorbent containing concentration w of adsorbate;
P,	defined by equation (8);
R,	defined by equation (8);
R_0,	defined by equation (10);
t,	time;
T,	temperature;
T_0,	constant temperature;
u^*,	potential in one phase in equilibrium with potential v in the other phase;
v,	potential in one phase;
w,	mass of adsorbate per unit mass of adsorbent;
w_0,	constant value of w;
x,	horizontal coordinate;
X,	dimensionless horizontal coordinate;
y,	vertical coordinate;
Y,	dimensionless vertical coordinate;
z,	linear coordinate.

Greek symbols

θ, dimensionless temperature [equation (11)];

θ_m, average value of θ [equation (36)];

θ_x, θ at intersection of operating line and equilibrium curve;

ρ, bulk density of adsorbent;

ϕ, dimensionless enthalpy [equation (27)];

ϕ^*, value of ϕ on equilibrium curve for given θ;

ϕ_0^*, value of ϕ on equilibrium curve for $\theta = 0$ [equation (28)].

INTRODUCTION

A LARGE part, perhaps the majority, of process heat- and mass-transfer calculations involve the estimation of the performance or the design of equipment for three different situations:

(a) Parallel flow of the phases;
(b) Cross flow of the phases;
(c) One stationary phase with the other phase moving through it.

To make the calculations a transfer coefficient is required and in this paper it is assumed to be known. We also require a relation describing the equilibrium between one phase and the other. In many cases this relation is linear, as for example in gas absorption, if Henry's law is obeyed, or when sensible heat is being transferred and the equilibrium temperature of one phase in terms of the temperature of the other phase is needed. Then the calculation for case (a) above can be made using a simple formula, while (b) and (c) can be solved with the aid of a chart containing families of curves, as long as the inlet or starting conditions of each phase is uniform.

Nevertheless many important industrial processes involve non-linear equilibrium relations. When cooling water with air, the equilibrium air enthalpy is a non-linear function of water temperature. When adsorbing a molecular species from a gas stream, the equilibrium partial pressure of the species is a non-linear function of the concentration of the species in the adsorbent. These relations, however, can often be expressed empirically and with suitable accuracy over large ranges by

$$u^* = A + Bv + C\exp(Dv) \tag{1}$$

and this is true of the particular and important processes just mentioned. u^* is the value of a concentration or potential in one phase which is in equilibrium with a concentration or potential v in the other phase. A, B, C and D are constants chosen to make a suitably accurate fit with equilibrium data. As an example, the equilibrium enthalpy of air, h^* Btu/lb dry air, is expressed with an accuracy better than 0·1 per cent in the range of water temperatures from 60°F to 90°F—the industrially important range—if $A = -10{\cdot}0$, $B = 0$, $C = \exp(1{\cdot}954)$ and $D = 0{\cdot}02352$ and v is the water temperature, T°F. A good fit over a larger range can be obtained if B is chosen to be non-zero.

This paper will show that, if the form of equation (1) can be accepted, charts can be prepared which will reduce the time required for calculations to the same as that required with linear equilibrium relations. The constants A, B, C and D would not appear as explicit variables so that the charts will be general in their application.

In the case of parallel flow two families of curves and hence basically two charts are required, one for concurrent and one for countercurrent flow. In practice, however, each chart would be broken into about ten fragments to obtain a high reading accuracy.

For the cases of cross flow and one stationary phase a set of charts would be required for each case to replace the single chart used with a linear equilibrium expression.

No attempt is made in this paper to present a comprehensive set of charts for obvious reasons of space limitation. It is emphasized however that, with an electronic computer, a set of about fifty charts can be produced which would remove the need for any further computer calculations.

The following sections derive the necessary expressions with reference to particular industrial processes. It will be clear, however, that the final expressions will be quite general.

PARALLEL FLOW

Consider a cooling tower operation. G is the mass flow rate of dry air with an enthalpy h per unit mass. L is the mass flow rate of water with a temperature T and specific heat c_L. L and G are both positive in concurrent flow. A heat balance is

$$c_L L\,\mathrm{d}T + G\,\mathrm{d}h = 0 \tag{2}$$

which integrates to give the "operating line"

$$c_L L(T - T_0) + G(h - h_0) = 0 \tag{3}$$

where subscript 0 refers to an arbitrary position in the tower to be chosen later.

The equilibrium line is

$$h^* = A + BT + C\exp(DT). \tag{4}$$

Now the calculation that has to be made is that for the number of transfer units (N.T.U.), defined by

$$\text{N.T.U.} = \int_{h_1}^{h_2} \frac{\mathrm{d}h}{h^* - h} = \int_{h_0}^{h_2} \frac{\mathrm{d}h}{h^* - h} - \int_{h_0}^{h_1} \frac{\mathrm{d}h}{h^* - h} = I_2 - I_1 \tag{5}$$

where

$$I = \int_{h_0}^{h} \frac{\mathrm{d}h}{h^* - h}. \tag{6}$$

Substitute equations (2), (3) and (4) in (6)

$$I = -\frac{P}{R}\int_{T_0}^{T} \frac{R\,\mathrm{d}T}{C\exp(DT) - H + R(T - T_0)} \tag{7}$$

where

$$H = [h_0 - A - BT_0], \qquad P = \frac{c_L L}{G},$$

$$R = (P + B). \tag{8}$$

Now to define new constants H_0 and R_0 and the reduced temperature θ write

$$H = H_0 C\exp(DT_0) \tag{9}$$

$$R = R_0 CD\exp(DT_0) \tag{10}$$

$$\theta = D(T - T_0) \tag{11}$$

when equation (7) becomes

$$I = -\frac{P}{R}\int_0^{\theta} \frac{R_0\,\mathrm{d}\theta}{\exp(\theta) - H_{00} \pm \theta}. \tag{12}$$

We can now make an arbitrary choice of a constant, which is equivalent to the choice of an arbitrary position in the tower, as mentioned above adjacent to equation (3). We choose $R_0 = \pm 1$ and equation (10) shows that the upper sign will apply to concurrent flow and the lower sign to countercurrent flow. It is noted that if D is negative C is also negative in practical cases. To emphasize that the choice has been made a second subscript 0 is employed. Thus equation (12) becomes

$$I = \mp\frac{P}{R}\int_0^{\theta} \frac{\mathrm{d}\theta}{\exp(\theta) - H_{00} \pm \theta}. \tag{13}$$

It is clear that $(R/P)I$ can be represented as a family of curves vs. θ with H_{00} as a parameter. To use the curves an equation for θ with H_{00} is required. From equations (10) and (11)

$$\theta = DT - \ln\left[\pm\frac{R}{CD}\right] \tag{14}$$

which is the equation for θ.

Divide equation (9) by (10)

$$H_{00} = \pm\frac{D}{R}H. \tag{15}$$

Substitution of the values of h_0 and T_0 from equations (3) and (11) into equation (8) gives

$$H = [h - A - BT] + \frac{R}{D}\theta. \tag{16}$$

From equations (15) and (16) we find the equation for H_{00} to be

$$H_{00} = \pm\frac{D}{R}[h - A - BT] \pm \theta. \quad (17)$$

Graphical presentation for parallel flow

Figure 1 shows the curve $\exp(\theta)$ vs. θ and four representative straight lines $(\theta + H_{00})$ vs. θ. It is clear from equation (13) and equation (6) that $\exp(\theta)$ is a reduced equilibrium curve and $(\theta + H_{00})$ is a reduced operating line for countercurrent flow. Figure 1 makes it plain that the extrapolated operating line will not intersect the equilibrium line if $H_{00} < 1$ since $e^{\theta} = 1$ and has a slope of unity at $\theta = 0$.

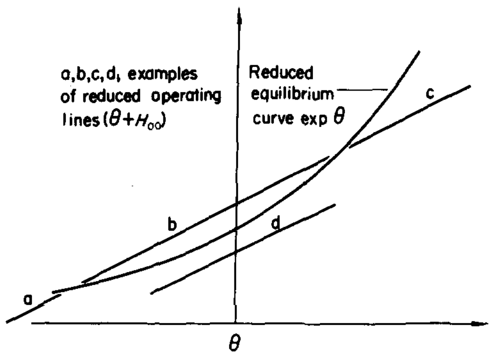

FIG. 1. Reduced equilibrium curve and operating lines in countercurrent flow.

The four representative operating lines of Fig. 1 indicate four situations encountered when evaluating $I(R/P)$ from equation (13). They form a suitable basis for fragmenting the whole family of curves into a number of charts. When the slope of the operating line is -1 we have, of course, only two different situations to consider.

An example of a chart for countercurrent flow is shown in Fig. 2. It is for a situation where the extrapolated operating line would intersect the equilibrium curve. As would be expected, the individual curves tend to infinity and become difficult to read with accuracy. This is of no concern since, where this happens, we can integrate equation (13) approximately with sufficient accuracy as follows. Find the point of intersection θ_x from

$$\exp(\theta_x) = \mp\theta_x + H_{00} \quad (18)$$

and write

$$\theta = \theta_x + \Delta\theta. \quad (19)$$

We find from equation (13) that if

$$\Delta\theta \leqslant 2[1 \pm \exp(-\theta_x)]$$

then

$$\frac{R}{P}[I_2 - I_1] \simeq \left[\frac{\mp 1}{\exp(\theta_x) \pm 1}\right] \ln\left[\frac{\Delta\theta_2}{\Delta\theta_1}\right]. \quad (20)$$

We can also find simple approximations where θ is very large, very small or close to zero.

Cross flow

Consider a crossflow cooling tower with the water falling in the positive y-direction and the air travelling horizontally in the x-direction. The heat balance is

$$\frac{\partial h}{\partial x} + \frac{c_L L}{G}\frac{\partial T}{\partial y} = 0. \quad (21)$$

The transfer equation is

$$G\frac{\partial h}{\partial x} = k(h^* - h) \quad (22)$$

where k is the transfer coefficient.

To get a useful result we must put $B = 0$ in the equilibrium relation. Thus

$$h^* = A + C\exp(DT). \quad (23)$$

The boundary conditions are

$$\text{at} \quad x = 0, \qquad h = h_0 \quad (24)$$

$$\text{at} \quad y = 0, \qquad T = T_0. \quad (25)$$

We define

$$\theta = D(T - T_0) \quad (26)$$

$$\phi = \frac{h - h_0}{C\exp(DT_0)} \quad (27)$$

$$\phi_0^* = \frac{h_0^* - h_0}{C\exp(DT_0)} \quad (28)$$

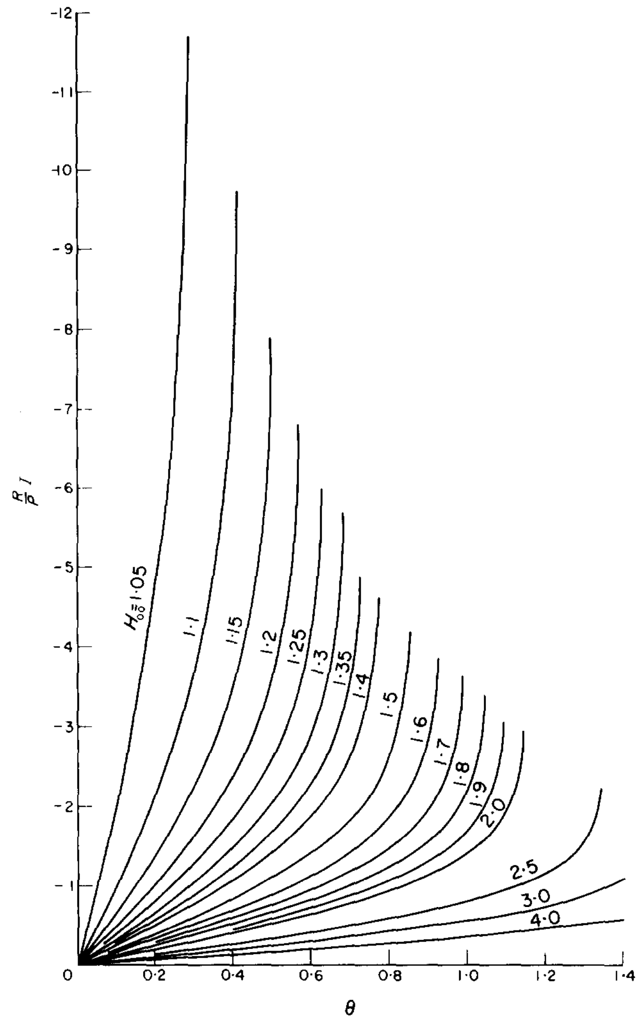

FIG. 2. An example of a chart for countercurrent flow. Equivalent to operating line *b* of Fig. 1.

$$X = \frac{k}{G}x \tag{29}$$

$$Y = \frac{kCD\exp(DT_0)}{c_L L}y \tag{30}$$

where

$$h_0^* = A + C\exp(DT_0). \tag{31}$$

Substituting equations (23), (26–30) in equations (21), (22), (24) and (25) we obtain

$$\frac{\partial\phi}{\partial X} + \frac{\partial\theta}{\partial Y} = 0 \tag{32}$$

$$\frac{\partial\phi}{\partial X} = \exp(\theta) - \phi + \phi_0^* - 1 \tag{33}$$

with the boundary conditions

$$\text{at} \quad X = 0 \qquad \phi = 0 \tag{34}$$

$$\text{at} \quad Y = 0 \qquad \theta = 0. \tag{35}$$

Equations (32) and (33) are solved without difficulty by a computer. An example of a resultant chart of particular use in cooling tower calculations is given in Fig. 3. In it lines of constant reduced mixed water temperature θ_m are plotted in X–Y coordinates for $\phi_0^* = 0{\cdot}5$ where

$$\theta_m = \left[\frac{1}{X}\int_0^X \theta \, \mathrm{d}X\right]_{Y=\text{constant.}} \tag{36}$$

The number of charts required for accurate cooling tower calculations is not more than ten.

The significance of different values of ϕ_0^* can be seen from Fig. 4 in which the equilibrium curve of equation (23) has been put in the reduced form

$$\phi^* = \phi_0^* + \exp(\theta) - 1 \tag{37}$$

and ϕ is plotted vs. θ. The origin of the graph represents the boundary conditions of (34) and (35). Conditions within the tower lie within the shaded area, which is approximately triangular. One side, however, is curved and extends from $\theta = 0$ to $\theta = \ln[1 - \phi_0^*]$. At $\theta = 0$ this side—the reduced equilibrium curve— has a slope of unity and at $\theta = \ln[1 - \phi_0^*]$ it has a slope of $[1 - \phi_0^*]$. Thus ϕ_0^* indicates the relative curvature of the equilibrium curve in the particular problem.

One stationary and one moving phase

Consider an unsteady state adsorption problem. There is a mass flow rate G of inert gas containing a weight m of adsorbate per unit weight of inert gas. The stationary phase has a

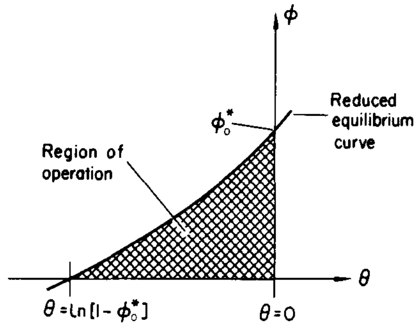

FIG. 4. Region of operation in crossflow.

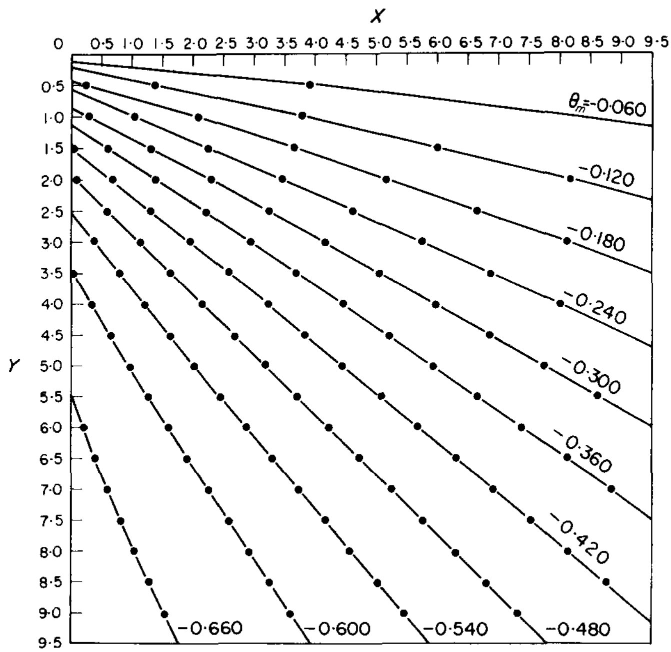

FIG. 3. Curves of constant θ_m in crossflow $\phi_0^* = 0{\cdot}5$.

bulk density ρ and a weight w of adsorbate per unit mass. The gas is travelling parallel to coordinate z and t is time. Then, ignoring the quantity of inert gas within the stationary phase, a mass balance gives

$$G\frac{\partial m}{\partial z} + \rho\frac{\partial w}{\partial t} = 0. \tag{38}$$

The transfer equation is

$$G\frac{\partial m}{\partial z} = k(m^* - m). \tag{39}$$

The equilibrium expression is

$$m^* = A + C\exp(Dw). \tag{40}$$

The boundary conditions are

$$\text{at} \quad z = 0 \qquad m = m_0 \tag{41}$$

$$\text{at} \quad t = 0 \qquad w = w_0. \tag{42}$$

These equations are exactly the same in form as equations (21–25) and thus an analogous treatment can be used but the dimensionless Y coordinate will become a dimensionless time coordinate. The form of the charts produced, however, may be different to satisfy the different objective of design calculations.

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Résumé—Des formules et des diagrammes simples sont déjà disponibles pour calculer les performances d'équipements de transport de chaleur et de masse si la relation exprimant l'équilibre entre les phases est linéaire et si l'on donne le coefficient de transport.

On donne comme exemples des gaz absorbants obéissant à la loi de Henry ou des échangeurs de chaleur transportant de la chaleur sensible.

On montre que des digrammes simples peuvent aussi être préparés dan lesquels la relation d'équilibre peut être exprimée sous la forme:

$$u^* = A + Bv + C\exp(Dv),$$

où u^* est un potentiel en équilibre avec le potentiel v dans une autre phase et A, B, C et D sont des constantes. Un nombre limité de diagrammes rend compte de toutes les conditions.

Les cas étudiés sont l'écoulement parallèle de phases—soit à courants parallèles, soit à contre-courant—l'écoulement perpendiculaire des phases et le cas transitoire pour lequel une phase est immobile et l'autre se déplace à travers elle.

La relation d'équilibre exponentielle peut être adaptée empiriquement avec une grande précision et dans des gammes étendues à de nombreux groupes importants au point de vue industriel de valeurs d'équilibre.

Zusammenfassung—Es existieren bereits einfache Formeln und Diagramme, um Wärme- und Stoff-Transport-Apparaturen zu berechnen, wenn die Beziehung zwischen den Phasen einem linearen Gesetz folgt und der Transport-Koeffizient gegeben ist.

Beispiele sind Gasabsorber, wenn das Henry'sche Gesetz gilt oder Wärmetauscher, die fühlbare Wärme übertragen.

$$u^* = A + Bv + C\exp(Dv)$$

hat, wobei u^* ein Potential im Gleichgewicht mit dem Potential v einer anderen Pjase darstellt und A, B, C und D Konstanten sind. Eine begrenzte Zahl von Diagrammen beschreibt alle Zusammenhänge vollständig.

Untersucht wurden die Anordnungen Gleichstrom, Gegenstrom, Kreuzstrom der Phasen und der zeitabhängige Fall, wobei eine Phase stationär ist und die andere durch sie bewegt wird.

Die exponentielle Gleichgewichtsbeziehung kann empirisch vielen industriell wichtigen Gruppen von Gleichgewichtsdaten mit grosser Genauigkeit und in weiten Bereichen angepasst werden.

Аннотация—Приводятся простые формулы и графики для расчета характеристик тепло-и массообменных установок при условии, что задан коэффициент переноса и отношения, описывающие равновесие между фазами, линейны.

В качестве примера рассматриваются газовые абсорберы для случая, если соблюдается закон Генри, и высокоинтенсивные теплообменники.

Показано, что можно построить простые графики, если представить уравнение равновесия в виде

$$u^* = A + Bv + C \exp(Dv)$$

где u^* есть потенциал, равновесный с потенциалом v в другой фазе, а A, B, C и D — константы. Небольшой объем графиков может отразить все случаи.

Рассматриваются случаи параллельного потока фаз, (прямотока и противотока), поперечного потока фаз и нестационарный случай, когда одна фаза неподвижна, а вторая проходит через неё.

Эмпирическое экспоненциальное уравнение равновесия можно с большой точностью применить в широком диапазоне ко многим практически важным классам данных по равновесию.